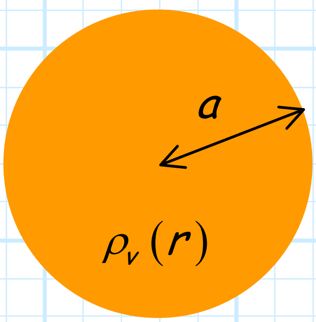
Example: Using Gauss's Law to Determine the Electric Field

Consider a "cloud" of charge with **radius** *a* and centered at the origin, described by volume charge density:



$$\rho_v(\bar{r}) = \begin{cases} \dfrac{1}{r} & r < a \\ 0 & r > a \end{cases}$$

Q: *What electric field* $\mathbf{E}(\bar{r})$ *is produced by this charge ?*

A: We **could** use Coloumb's Law to solve this, but note that this is a **spherically symmetric** charge density! As a result, we can find the electric field much easier using **Gauss's Law**.

Recall that **spherically symmetric** charge densities produce an electric field:

$$\mathbf{E}(\bar{\mathbf{r}}) = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}\,\hat{a}_r$$

$$= \frac{\hat{a}_r}{\varepsilon_0 r^2}\int_0^r \rho_v(r')\,r'^2 dr'$$

Evaluating the **integral**, we need to consider **two** cases: one where r (i.e., the radius of the **Gaussian surface**) is **less than** cloud radius a (for evaluating the field **within** the charge cloud), and the second where r is **greater than** cloud radius a (for evaluating the field **outside** the charge cloud).

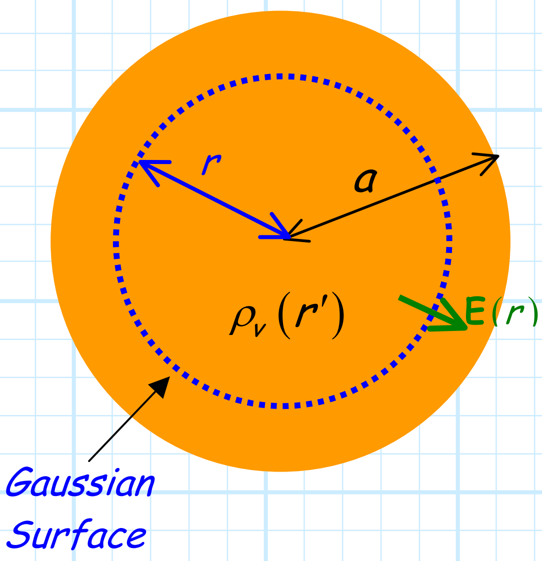
For $r < a$:

$$\int_0^r \rho_v(r')\,r'^2 dr' = \int_0^r \left(\frac{1}{r'}\right) r'^2 dr'$$

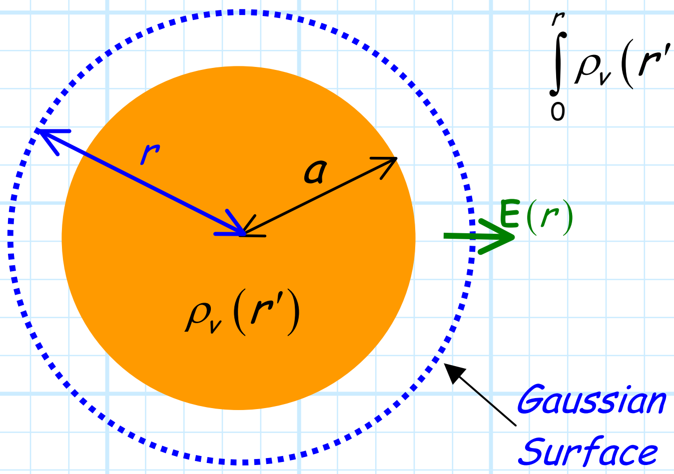
$$= \int_0^r r' dr'$$

$$= \left.\frac{r'^2}{2}\right|_0^r$$

$$= \frac{r^2}{2}$$



And if **r** > *a*:



$$\begin{aligned}\int_0^r \rho_v(r')\, r'^2 dr' &= \int_0^a \left(\frac{1}{r'}\right) r'^2 dr' + \int_a^r 0\, r'^2 dr' \\ &= \int_0^a r' dr' + 0 \\ &= \left.\frac{r'^2}{2}\right|_0^a \\ &= \frac{a^2}{2}\end{aligned}$$

Therefore, the **electric field** produced by this charge is:

$$\begin{aligned}\mathbf{E}(\bar{\mathrm{r}}) &= \frac{\hat{a}_r}{\varepsilon_0 r^2}\int_0^r \rho_v(r')\, r'^2 dr' \\ &= \begin{cases} \dfrac{\hat{a}_r}{\varepsilon_0 r^2}\left(\dfrac{r^2}{2}\right) & r < a \\[2ex] \dfrac{\hat{a}_r}{\varepsilon_0 r^2}\left(\dfrac{a^2}{2}\right) & r > a \end{cases} \\ &= \begin{cases} \dfrac{\hat{a}_r}{2\varepsilon_0} & r < a \\[2ex] \dfrac{\hat{a}_r}{2\varepsilon_0}\left(\dfrac{a^2}{r^2}\right) & r > a \end{cases}\end{aligned}$$

Note the resulting electric field behaves **as expected**. The field points in the direction $\hat{a}_r$ (i.e., points away from the origin). It is likewise independent of θ or ϕ (i.e., **spherically symmetric**).

Note also that the magnitude of the field outside of the cloud **diminishes** as $1/r^2$. This **makes sense**! Do **you** see why?